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Construction of exact dynamical invariants in coupled oscillator problems

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Abstract

With a view to obtaining further insight into the theoretical understanding of the problem of coupled harmonic oscillators we carry out the construction of exact dynamical invariants for momentum- and time-dependent (TD) Hamiltonian systems in two dimensions. In particular, we investigate the systems

$$H_1 = \tfrac{1}{2}[\alpha_1 p_1^2 + \alpha_2 p_2^2 + \beta_1 x_1^2 + \beta_2 x_2^2 + 2\beta_3 x_1 x_2 + 2\alpha_3 p_1 p_2]$$
$$H_2 = \tfrac{1}{2}\alpha(p_1^2 + p_2^2) + \tfrac{1}{2}\beta(x_1^2 + x_2^2) + f(p_1 x_2 - p_2 x_1)$$

where the parameters $\alpha_i, \beta_i, i = 1, 2, 3, \alpha, \beta, f$ may be TD. While the Lie algebraic method is employed for the TD forms of H_1 and H_2, the rationalization method, modified here for the momentum-dependent case, is used for the time-independent versions of H_1 and H_2. The role and scope of the invariants so constructed is pointed out.

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1. Introduction

Invariants of a dynamical system, if they exist and become available, can prove [1–3] to be an asset as far as the theoretical understanding of the system is concerned. The search for these dynamical invariants for one- and higher-dimensional systems in terms of the methods of their construction has been [4] going on for a long time now. Also, in recent years, the explicit time variable dependence of the underlying parameters of the system has suggested another line of study for these systems [3, 5]. It may be mentioned that these studies, dealing mainly with the construction and use of exact [3] invariants, have only been carried out for a few systems. A large number of dynamical systems in the literature have been studied using the so-called approximation or perturbation methods [6, 7] and accordingly one deals with the 'approximate' invariants or the 'first integrals' of motion for the system. Here, however, we shall restrict ourselves to the study of exact invariants. Furthermore, while the coupled harmonic oscillator

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problem in two dimensions offers an example in this general scheme of study, its variants with reference to the singular coupling terms has again been [3,5] the subject of research for the last three decades or so under the heading of generalized Ermakov (or Lewis) systems, particularly for the time-dependent (TD) cases.

In this paper we focus our attention on the study of two-dimensional systems described respectively by the Hamiltonians

$$H_1 = \tfrac{1}{2}(\alpha_1 p_1^2 + \alpha_2 p_2^2) + \tfrac{1}{2}(\beta_1 x_1^2 + \beta_2 x_2^2) + \beta_3 x_1 x_2 + \alpha_3 p_1 p_2 \tag{1}$$

and

$$H_2 = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) + f(p_1 x_2 - p_2 x_1) \tag{2}$$

in which the coupling terms not only involve the dependence on the momenta p_1 and p_2 but the parameters α_i, β_i, $i = 1, 2, 3, m, \omega^2$ and f could also be TD. While we carry out the construction of dynamical invariants for the TD cases of H_1 and H_2 specifically using the Lie algebraic approach [3], the invariants for the time-independent (TID) cases will be obtained using both rationalization and Lie algebraic methods. As the coupling terms in (1) now depend on the momenta p_1 and p_2, we modify the existing [2,3] rationalization method for this purpose in section 2.

With regard to the momentum-dependent coupling in the Hamiltonian, such a situation is encountered in a variety of physical situations, namely in the description of motion of a charged particle in a magnetic field [8,9] or in the Bell's inequality experiments employing [10] four coupled harmonic oscillators. In fact, in the latter case the Hamiltonian H_1 of (1) is a physical realization of the interaction Hamiltonian of the form $H_{\rm int} = k(x_1 x_2 + p_1 p_2)$, where $k = \alpha_3 = \beta_3$ is the coupling constant. A coupling of the form kx_1x_2 is known in a mechanical context and it has a relation with the energy of two mechanical oscillators (the energy of a spring connected between two mechanical oscillators, in general, is of the form $\frac{1}{2}k'(x_1 - x_2)^2$ and the same gives rise to the coupling $-k'x_1x_2$ in the Hamiltonian). On the other hand, the coupling of the form kp_1p_2, though unusual in the mechanical context, can appear [9] in the context of electrical circuits where the charge plays the same role as position and the current as velocity in mechanical systems. In electrical circuits, however, the coupling kp_1p_2 is recognized as the mutual inductance term in the Hamiltonian [9, 10],

$$H = \frac{1}{2}L_1\left(\frac{\mathrm{d}Q_1}{\mathrm{d}t}\right)^2 + \frac{1}{2}\left(\frac{1}{C_1} + \frac{1}{C_M}\right)Q_1^2 + \frac{1}{2}L_2\left(\frac{\mathrm{d}Q_2}{\mathrm{d}t}\right)^2 + \frac{1}{2}\left(\frac{1}{C_2} + \frac{1}{C_M}\right)Q_2^2 + \frac{1}{C_M}Q_1Q_2 + M\left(\frac{\mathrm{d}Q_1}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}Q_2}{\mathrm{d}t}\right) \tag{3}$$

which corresponds to a pair of coupled inductance–capacitance (LC) circuits, namely, when the L_1C_1 and L_2C_2 circuits are coupled through the capacitor C_M. Furthermore, the coupling of inductances L_1 and L_2 contributes to the mutual inductance M. Moreover, the study of electrical circuits with time-varying capacitors and inductors, particularly with reference to their memory property, has become [11] of considerable interest in recent years.

Recall that the nonlocal potentials used in nuclear physics to study the binding energy of nuclei are also sometimes considered as momentum dependent. Moreover, the interaction $k(x_1x_2+p_1p_2)$ performs the same function as a beam splitter does in an optical experiment [10]. Very recently 't Hooft [12] and Blasone *et al* [13] studied momentum-dependent terms in the Hamiltonian structure in the context of the so-called holographic principle and in the treatment of quantum gravity as a dissipative and deterministic system. The study of the Hamiltonian (2) is needed [14] in a variety of situations. For example, H_2 of equation (2) is used in the study of various phenomena at the quantum level, namely to study [15] the quantum motion of a particle in a Paul trap, to describe [16] a quantized electromagnetic field in a Fabry–Parot cavity, to control the atoms [17] by means of laser beams or other electric and magnetic fields [18], etc.

Before proceeding further we briefly outline here the essential steps of the Lie algebraic method which has been employed [3, 5, 19] in the past for a variety of dynamical systems. In this method, one expresses the two-dimensional TD Hamiltonian in the form

$$H = \sum_i h_i(t)\Gamma_i(x_1, x_2, p_1, p_2). \tag{4}$$

Here h_i are the TD coefficients and Γ_i are the phase space functions which are required to close the algebra with respect to the Poisson bracket,

$$[\Gamma_i, \Gamma_j] = \sum_k C_{ij}^k \Gamma_k \tag{5}$$

with C_{ij}^k as the structure constants of the Lie algebra. Further, for the two-dimensional case the Poisson bracket is defined as

$$[\mathcal{A}, \mathcal{B}] = \frac{\partial\mathcal{A}}{\partial x_1}\frac{\partial\mathcal{B}}{\partial p_1} - \frac{\partial\mathcal{A}}{\partial p_1}\frac{\partial\mathcal{B}}{\partial x_1} + \frac{\partial\mathcal{A}}{\partial x_2}\frac{\partial\mathcal{B}}{\partial p_2} - \frac{\partial\mathcal{A}}{\partial p_2}\frac{\partial\mathcal{B}}{\partial x_2}. \tag{6}$$

For the purposes of closing the algebra through (5) one might need some additional Γ_l which can be introduced by setting the corresponding coefficients $h_l(t) = 0$ in (4). Since the invariant I is also a member of the dynamical algebra, the same can be expressed as

$$I = \sum_m \lambda_m(t)\Gamma_m(x_1, x_2, p_1, p_2) \tag{7}$$

which also satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + [I, H] = 0. \tag{8}$$

Now using (4) and (7) in (8) and subsequently rationalizing the resultant expression after using (5), one arrives [3, 19] at a set of first-order, coupled differential equations, namely,

$$\dot{\lambda}_k + \sum_i \left[\sum_j C_{ij}^k h_j(t)\right]\lambda_i = 0. \tag{9}$$

These equations can be solved for λ_k and their substitution in (7) leads to the required invariant I.

In section 2, we construct the second invariant for the TID versions of H_1 and H_2 (as H_1 and H_2 themselves represent constants of motion of the system concerned). In section 3, we make use of the Lie algebraic approach to derive at least one invariant for the TD versions of each H_1 and H_2. In fact, the Lie algebraic approach commands [3] several advantages over the rationalization method, particularly for the TD systems, not only in terms of the closure property of the Poisson bracket algebra of phase space functions but also for its straightforward extension to the corresponding [20, 21] quantum system. Finally, concluding remarks are given in section 4 by highlighting the possible role of these constructed invariants in different physical situations.

2. Second invariant for momentum-dependent potentials

2.1. General results

While the Lie algebraic approach for the construction of dynamical invariants automatically takes care of the momentum dependence of the system through the Poisson bracket algebra of phase space functions, its use, however, becomes complicated at the level of the closure property of the algebra for a large number of systems. Therefore, one resorts to using the conventional rationalization method for this purpose. For a two-dimensional momentum-dependent Hamiltonian of the form

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + A(x_1, x_2)p_1 + B(x_1, x_2)p_2 + W(x_1, x_2). \tag{10}$$

Dorizzi *et al* [22] have discussed the construction of the second invariant. Since the system (10) can easily be identified with (2) but not with (1), we proceed here to investigate a general form of the momentum-dependent two-dimensional Hamiltonian of the form

$$H(x_1, x_2, p_1, p_2) = \tfrac{1}{2}(\alpha_1 p_1^2 + \alpha_2 p_2^2) + V(x_1, x_2, p_1, p_2) \tag{11}$$

for which Hamilton's equations of motion can be written as

$$\overset{\circ}{x}_1 = \alpha_1 p_1 + \frac{\partial V}{\partial p_1} \qquad \overset{\circ}{p}_1 = -\frac{\partial V}{\partial x_1} \tag{12a}$$

$$\overset{\circ}{x}_2 = \alpha_2 p_2 + \frac{\partial V}{\partial p_2} \qquad \overset{\circ}{p}_2 = -\frac{\partial V}{\partial x_2} \tag{12b}$$

or, in a more compact form these equations of motion can be expressed as

$$\overset{\circ\circ}{x}_i \equiv \overset{\circ}{\xi}_i = v_i + \frac{\partial^2 V}{\partial x_j \partial p_i}\xi_j \tag{13}$$

where $\xi_i = \overset{\circ}{x}_i$, $i = 1, 2$, and

$$v_1 = -\left[\left(\alpha_1 + \frac{\partial^2 V}{\partial p_1^2}\right)\frac{\partial V}{\partial x_1} + \frac{\partial^2 V}{\partial p_2 \partial p_1}\cdot\frac{\partial V}{\partial x_2}\right] \tag{14a}$$

$$v_2 = -\left[\left(\alpha_2 + \frac{\partial^2 V}{\partial p_2^2}\right)\frac{\partial V}{\partial x_2} + \frac{\partial^2 V}{\partial p_1 \partial p_2}\cdot\frac{\partial V}{\partial x_1}\right]. \tag{14b}$$

Next, for the second invariant I of the system (11) up to second order in the momenta, we make an ansatz of the form

$$I = a_0 + a_i\xi_i + \tfrac{1}{2}a_{ij}\xi_i\xi_j \tag{15}$$

where $i, j = 1, 2$ and the coefficient functions a_0, a_i and a_{ij} are functions of x_1 and x_2 with $a_{ij} = a_{ji}$. The fact that I is an invariant of the system (11) requires

$$\frac{\mathrm{d}I}{\mathrm{d}t} = [I, H] = 0 \tag{16a}$$

or

$$a_{0,i}\xi_i + a_{i,j}\xi_i\xi_j + a_i\overset{\circ}{\xi}_i + \tfrac{1}{2}a_{ij,k}\xi_i\xi_j\xi_k + \tfrac{1}{2}a_{ij}(\overset{\circ}{\xi}_i\xi_j + \xi_i\overset{\circ}{\xi}_j) = 0 \tag{16b}$$

which, after using (13) for $\overset{\circ}{\xi}_i$ and rationalizing the resultant expression with respect to the products of ξ_i, yields [3] the following equations:

$$a_i v_i = 0 \tag{17}$$

$$a_{0,i} + a_{ij}v_j + \frac{\partial^2 V}{\partial x_i \partial p_j}a_j = 0 \tag{18}$$

$$a_{i,j} + a_{j,i} + a_{kj}\frac{\partial^2 V}{\partial x_i \partial p_k} + a_{ik}\frac{\partial^2 V}{\partial x_j \partial p_k} = 0 \tag{19}$$

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0. \tag{20}$$

Note that while the solution of (20) can be obtained (cf [3], ch 2) immediately as

$$\begin{aligned} &a_{11} = \tfrac{1}{2}c_1x_2^2 + c_2x_2 + c_3 \qquad a_{22} = \tfrac{1}{2}c_1x_1^2 + c_4x_1 + c_5 \\ &a_{12} = -\tfrac{1}{2}(c_1x_1x_2 + c_2x_1 + c_4x_2 - 2c_6) \end{aligned} \tag{21}$$

where c_i, $i = 1, \ldots, 6$, are arbitrary constants of integration, the expanded versions of equations (17)–(19) are given by

$$a_1 v_1 + a_2 v_2 = 0 \tag{22}$$

$$\frac{\partial a_0}{\partial x_1} + a_{11} v_1 + a_{12} v_2 + a_1 \frac{\partial^2 V}{\partial x_1 \partial p_1} + a_2 \frac{\partial^2 V}{\partial x_1 \partial p_2} = 0 \tag{23a}$$

$$\frac{\partial a_0}{\partial x_2} + a_{21} v_1 + a_{22} v_2 + a_1 \frac{\partial^2 V}{\partial x_2 \partial p_1} + a_2 \frac{\partial^2 V}{\partial x_2 \partial p_2} = 0 \tag{23b}$$

$$\frac{\partial a_1}{\partial x_1} + a_{11} \frac{\partial^2 V}{\partial x_1 \partial p_1} + a_{21} \frac{\partial^2 V}{\partial x_1 \partial p_2} = 0 \tag{24a}$$

$$\frac{\partial a_2}{\partial x_2} + a_{22} \frac{\partial^2 V}{\partial x_2 \partial p_2} + a_{12} \frac{\partial^2 V}{\partial x_2 \partial p_1} = 0 \tag{24b}$$

$$\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_1} + a_{11} \frac{\partial^2 V}{\partial x_2 \partial p_1} + a_{22} \frac{\partial^2 V}{\partial x_1 \partial p_2} + a_{12} \left(\frac{\partial^2 V}{\partial x_2 \partial p_2} + \frac{\partial^2 V}{\partial x_1 \partial p_1} \right) = 0. \tag{24c}$$

Now we look for the solutions of equations (22)–(24*c*) for a given potential V. As a matter of fact these equations represent an overdetermined system of equations to find the coefficient functions a_0, a_1, a_2. What we shall give below are some sort of constraining relations to be satisfied by the given potential. For simplicity we assume, in accordance with (22), that the coefficient functions a_1 and a_2 are given by

$$a_1 = v_2 \qquad a_2 = -v_1. \tag{25}$$

This immediately leads to three constraining relations from equations (24*a*)–(24*c*), namely

$$\frac{\partial v_2}{\partial x_1} + a_{11} \frac{\partial^2 V}{\partial x_1 \partial p_1} + a_{21} \frac{\partial^2 V}{\partial x_1 \partial p_2} = 0 \tag{26}$$

$$\frac{\partial v_1}{\partial x_2} - a_{22} \frac{\partial^2 V}{\partial x_2 \partial p_2} - a_{12} \frac{\partial^2 V}{\partial x_2 \partial p_1} = 0 \tag{27}$$

$$\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_1} + a_{11} \frac{\partial^2 V}{\partial x_2 \partial p_1} + a_{22} \frac{\partial^2 V}{\partial x_1 \partial p_2} + a_{12} \left(\frac{\partial^2 V}{\partial x_2 \partial p_2} + \frac{\partial^2 V}{\partial x_1 \partial p_1} \right) = 0 \tag{28}$$

which are termed [3] as 'potential' equations in the sense that their solutions would directly provide the potential function $V(x_1, x_2, p_1, p_2)$ that admit the second-order invariant (15). Also, these relations can be used to determine the arbitrary c_i constants appearing in (21). Thus, the coefficient functions a_{ij} and a_i are given by equations (21) and (25), a_0 from (23*a*) and (23*b*) and finally using these results the invariant I can be derived immediately from (15) for the system (11).

It may be mentioned that the potential equations (26)–(28) are obtained after using the assumption (25) for the solution of (22). This setting, however, does not always work, particularly when the functions v_1 and v_2 depend on the momenta p_1 and p_2 (cf system H_2 of equation (2)). In that case one has to rationalize all six equations (22)–(24*c*) to determine not only the unknown c_i appearing in equation (21) but also the remaining coefficient functions a_1, a_2 and a_0.

In what follows we apply these general results to some specific forms of $V(x_1, x_2, p_1, p_2)$, particularly the ones appearing in the systems H_1 and H_2.

2.2. Applications

In order to demonstrate the applications of the general prescription of section 2.1, we consider here the potential functions appearing in systems (1) and (2).

Case I. Corresponding to system (1) consider the momentum-dependent potential

$$V(x_1, x_2, p_1, p_2) = \tfrac{1}{2}(\beta_1 x_1^2 + \beta_2 x_2^2 + 2\beta_3 x_1 x_2) + \alpha_3 p_1 p_2. \tag{29}$$

For this simple case, v_1 and v_2 can be obtained from (14) as

$$\begin{aligned} v_1 &= -[(\alpha_1\beta_1 + \alpha_3\beta_3)x_1 + (\alpha_1\beta_3 + \alpha_3\beta_2)x_2] \\ v_2 &= -[(\alpha_2\beta_3 + \alpha_3\beta_1)x_1 + (\alpha_2\beta_2 + \alpha_3\beta_3)x_2]. \end{aligned} \tag{30}$$

Use of (30) and (21) in the potential equations (26)–(28) immediately yields the constraining relations

$$\alpha_2\beta_3 + \alpha_3\beta_1 = 0 \tag{31a}$$

$$\alpha_1\beta_3 + \alpha_3\beta_2 = 0 \tag{31b}$$

$$\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_0\beta_0 \quad \text{(say)} \tag{31c}$$

on the potential parameters in (29). In fact only two of these conditions are independent. In the mechanical context, here we shall consider the case when the masses of the coupled oscillators are proportional to the squares of the corresponding frequencies as dictated by the condition (31*c*). On the other hand, in the electrical context (cf Hamiltonian (3)) this condition is tantamount to the fact that the frequencies are inversely proportional to the respective inductances. These restrictions give rise to the forms of v_1 and v_2 from (30) as $v_1 = -A_0 x_1$, $v_2 = -A_0 x_2$ with $A_0 = \alpha_0\beta_0 + \alpha_3\beta_3$. Furthermore, after using these results equations (23*a*) and (23*b*) can be integrated to yield a unique expression for $a_0(x_1, x_2)$, namely

$$a_0(x_1, x_2) = A_0(\tfrac{1}{2}c_3 x_1^2 + \tfrac{1}{2}c_5 x_2^2 + c_6 x_1 x_2) \tag{32}$$

and it will also set simultaneously $c_2 = c_4 = 0$. Thus, for other coefficient functions we obtain

$$\begin{aligned} &a_1 = -A_0 x_2 \qquad a_2 = A_0 x_1 \qquad a_{11} = \tfrac{1}{2}c_1 x_2^2 + c_3 \\ &a_{22} = \tfrac{1}{2}c_1 x_1^2 + c_5 \qquad a_{12} = -\tfrac{1}{2}(c_1 x_1 x_2 - 2c_6). \end{aligned} \tag{33}$$

Finally, the invariant I can be obtained from (15) as

$$\begin{aligned} I = A_0[&\tfrac{1}{2}c_3 x_1^2 + \tfrac{1}{2}c_5 x_2^2 + c_6 x_1 x_2 + \alpha_2 x_1 p_2 - \alpha_1 p_1 x_2 + \alpha_3(p_1 x_1 - p_2 x_2)] \\ &+ \tfrac{1}{4}c_1[\alpha_2 x_1 p_2 - \alpha_1 p_1 x_2 + \alpha_3(p_1 x_1 - p_2 x_2)]^2 + \tfrac{1}{2}[c_3(\alpha_1 p_1 + \alpha_3 p_2)^2 \\ &+ c_5(\alpha_2 p_2 + \alpha_3 p_1)^2] + c_6(\alpha_1 p_1 + \alpha_3 p_2)(\alpha_2 p_2 + \alpha_3 p_1) \end{aligned} \tag{34}$$

where $\xi_1 \equiv \overset{\circ}{x}_1 = \alpha_1 p_1 + \alpha_3 p_2$ and $\xi_2 \equiv \overset{\circ}{x}_2 = \alpha_2 p_2 + \alpha_3 p_1$ are used from equations (12).

Note that for the TID version of system (1) it is also possible to derive an invariant in a heuristic manner, i.e. without going through the formal procedure of section 2.1. In fact it can be seen that for the system (1), Hamilton's equations of motion turn out to be

$$\overset{\circ}{x}_1 = \frac{\partial H_1}{\partial p_1} = \alpha_1 p_1 + \alpha_3 p_2 \qquad \overset{\circ}{p}_1 = -\frac{\partial H_1}{\partial x_1} = -(\beta_1 x_1 + \beta_3 x_2) \tag{35}$$

$$\overset{\circ}{x}_2 = \frac{\partial H_1}{\partial p_2} = \alpha_2 p_2 + \alpha_3 p_1 \qquad \overset{\circ}{p}_2 = -\frac{\partial H_1}{\partial x_2} = -(\beta_2 x_2 + \beta_3 x_1). \tag{36}$$

These equations can be expressed as

$$\overset{\circ\circ}{x}_1 = -(\alpha_1\beta_1 + \alpha_3\beta_3)x_1 - (\alpha_1\beta_3 + \alpha_3\beta_2)x_2 \tag{37}$$

$$\overset{\circ\circ}{x}_2 = -(\alpha_2\beta_3 + \alpha_3\beta_1)x_1 - (\alpha_2\beta_2 + \alpha_3\beta_3)x_2. \tag{38}$$

Now, after multiplying equation (38) by $\overset{\circ}{x}_1$ and (37) by $\overset{\circ}{x}_2$ and adding the resultant equations we arrive at

$$\begin{aligned} \overset{\circ}{x}_1\overset{\circ\circ}{x}_2 + \overset{\circ\circ}{x}_1\overset{\circ}{x}_2 = -[&(\alpha_2\beta_3 + \alpha_3\beta_1)\overset{\circ}{x}_1 x_1 + (\alpha_2\beta_2 + \alpha_3\beta_3)\overset{\circ}{x}_1 x_2 \\ &+ (\alpha_1\beta_1 + \alpha_3\beta_3)x_1\overset{\circ}{x}_2 + (\alpha_1\beta_3 + \alpha_3\beta_2)x_2\overset{\circ}{x}_2]. \end{aligned} \tag{39}$$

For the case when $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_0\beta_0$ (say), expression (39) can be immediately integrated to yield the invariant of the system (1) in the form

$$I = (\alpha_1 p_1 + \alpha_3 p_2)(\alpha_2 p_2 + \alpha_3 p_1) + \tfrac{1}{2}[(\alpha_2\beta_3 + \alpha_3\beta_1)x_1^2 + (\alpha_1\beta_3 + \alpha_3\beta_2)x_2^2 + 2(\alpha_0\beta_0 + \alpha_3\beta_3)x_1x_2] \tag{40}$$

where $\overset{\circ}{x}_1 = \alpha_1 p_1 + \alpha_3 p_2$, $\overset{\circ}{x}_2 = \alpha_2 p_2 + \alpha_3 p_1$ is used. Note that while both the forms (34) and (40) of the invariant I conform to the requirement (16*a*), the form (40) is independent of any arbitrary constant such as c_1, c_3, c_5 or c_6. Moreover, the form (40) exists with one restriction on the potential parameters only, namely $\alpha_1\beta_1 = \alpha_2\beta_2$, whereas for the existence of (34) we have two restrictions (cf equations (31)). For the integrability of the TID two-dimensional system (1) one expects the existence of one more invariant (say, equation (34)) besides the Hamiltonian (1). The existence of the additional invariant (40) also for the same system (1), however, indicates the superintegrability [2, 3] of the system (1).

Case II. After identifying $\alpha = (1/m)$, $\beta = m\omega^2$, the potential function of system (2) can be written as

$$V(x_1, x_2, p_1, p_2) = \tfrac{1}{2}\beta(x_1^2 + x_2^2) + f(p_1x_2 - p_2x_2) \tag{41}$$

and for this case note from (8) that v_1 and v_2, namely

$$v_1 = -\alpha(\beta x_1 - f p_2) \qquad v_2 = -\alpha(\beta x_2 + f p_1) \tag{42}$$

turn out to be momentum-dependent functions in contrast to assumption (25). Therefore, the construction of the invariant using the rationalization method in this case becomes rather involved. For system (2), when it is partitioned in the form

$$H_2 = H_2^{(1)} + H_2^{(2)} \tag{43}$$

where

$$H_2^{(1)} = \tfrac{1}{2}\alpha(p_1^2 + p_2^2) + \tfrac{1}{2}\beta(x_1^2 + x_2^2) \qquad H_2^{(2)} = f(p_1x_2 - p_2x_1) \tag{44}$$

one can note that each $H_2^{(1)}$ and $H_2^{(2)}$, separately, are the constants of motion of the system since $H_2^{(1)}$ represents the Hamiltonian for a pair of decoupled oscillators and $H_2^{(2)}$ is the third component of the angular momentum. However, we shall carry out some nontrivial constructions for this case in the next section using the Lie algebraic method.

3. Time-dependent invariants

Here, using the Lie algebraic approach outlined in section 1, we derive the invariants for the case when the parameters in systems (1) and (2) are TD.

3.1. Invariant for H_1

In order to express H_1 of (1) in the form (4) we make the following identifications for the phase space functions Γ_i and the coefficient functions h_i:

$$\begin{array}{lll} \Gamma_1 = \tfrac{1}{2}p_1^2 & \Gamma_2 = \tfrac{1}{2}p_2^2 & \Gamma_3 = p_1p_2 \\ \Gamma_4 = \tfrac{1}{2}x_1^2 & \Gamma_5 = \tfrac{1}{2}x_2^2 & \Gamma_6 = x_1x_2 \end{array} \tag{45a}$$

$$\begin{array}{lll} h_1 = \alpha_1(t) & h_2 = \alpha_2(t) & h_3 = \alpha_3(t) \\ h_4 = \beta_1(t) & h_5 = \beta_2(t) & h_6 = \beta_3(t). \end{array} \tag{45b}$$

Further note that the closure of the dynamical algebra requires four more Γ_l in this case, namely $\Gamma_7 = p_1x_1$; $\Gamma_8 = p_1x_2$; $\Gamma_9 = p_2x_2$, $\Gamma_{10} = p_2x_1$ with the corresponding h_l as $h_7 = h_8 = h_9 = h_{10} = 0$. Also, the nonvanishing Poisson brackets in (8) computed using (6), now turn out to be

$$
\begin{array}{lll}
[\Gamma_1, \Gamma_4] = -\Gamma_7 & [\Gamma_1, \Gamma_6] = -\Gamma_8 & [\Gamma_1, \Gamma_7] = -2\Gamma_1 \\
[\Gamma_1, \Gamma_{10}] = -\Gamma_3 & [\Gamma_2, \Gamma_5] = -\Gamma_9 & [\Gamma_2, \Gamma_6] = -\Gamma_{10} \\
[\Gamma_2, \Gamma_8] = -\Gamma_3 & [\Gamma_2, \Gamma_9] = -2\Gamma_2 & [\Gamma_3, \Gamma_4] = -\Gamma_{10} \\
[\Gamma_3, \Gamma_5] = -\Gamma_8 & [\Gamma_3, \Gamma_6] = -\Gamma_7 - \Gamma_9 & \\
[\Gamma_3, \Gamma_7] = -\Gamma_3 & [\Gamma_3, \Gamma_8] = -2\Gamma_1 & [\Gamma_3, \Gamma_9] = -\Gamma_3 \\
[\Gamma_3, \Gamma_{10}] = -2\Gamma_2 & [\Gamma_4, \Gamma_7] = 2\Gamma_4 & \\
[\Gamma_4, \Gamma_8] = \Gamma_6 & [\Gamma_5, \Gamma_9] = 2\Gamma_5 & [\Gamma_5, \Gamma_{10}] = \Gamma_6 \\
[\Gamma_6, \Gamma_7] = \Gamma_6 & [\Gamma_6, \Gamma_8] = 2\Gamma_5 & \\
[\Gamma_6, \Gamma_9] = \Gamma_6 & [\Gamma_6, \Gamma_{10}] = 2\Gamma_4 & [\Gamma_7, \Gamma_8] = \Gamma_8 \\
[\Gamma_7, \Gamma_{10}] = -\Gamma_{10} & [\Gamma_8, \Gamma_9] = \Gamma_8 & \\
[\Gamma_8, \Gamma_{10}] = \Gamma_7 - \Gamma_9 & [\Gamma_9, \Gamma_{10}] = \Gamma_{10}. &
\end{array}
\tag{46}
$$

Substitution of these results in (8) and the subsequent rationalization of the resultant expression with respect to Γ_i yield the following set of differential equations for λ_k:

$$\overset{\circ}{\lambda}_1 = -2\alpha_1\lambda_7 - 2\alpha_3\lambda_8 \tag{47a}$$

$$\overset{\circ}{\lambda}_2 = -2\alpha_2\lambda_9 - 2\alpha_3\lambda_{10} \tag{47b}$$

$$\overset{\circ}{\lambda}_3 = -\alpha_3\lambda_7 - \alpha_2\lambda_8 - \alpha_3\lambda_9 - \alpha_1\lambda_{10} \tag{47c}$$

$$\overset{\circ}{\lambda}_4 = 2\beta_1\lambda_7 + 2\beta_3\lambda_{10} \tag{47d}$$

$$\overset{\circ}{\lambda}_5 = 2\beta_3\lambda_8 + 2\beta_2\lambda_9 \tag{47e}$$

$$\overset{\circ}{\lambda}_6 = \beta_3\lambda_7 + \beta_1\lambda_8 + \beta_3\lambda_9 + \beta_2\lambda_{10} \tag{47f}$$

$$\overset{\circ}{\lambda}_7 = \beta_1\lambda_1 + \beta_3\lambda_3 - \alpha_1\lambda_4 - \alpha_3\lambda_6 \tag{47g}$$

$$\overset{\circ}{\lambda}_8 = \beta_3\lambda_1 + \beta_2\lambda_3 - \alpha_3\lambda_5 - \alpha_1\lambda_6 \tag{47h}$$

$$\overset{\circ}{\lambda}_9 = \beta_2\lambda_2 + \beta_3\lambda_3 - \alpha_3\lambda_6 - \alpha_2\lambda_5 \tag{47i}$$

$$\overset{\circ}{\lambda}_{10} = \beta_3\lambda_2 + \beta_1\lambda_3 - \alpha_3\lambda_4 - \alpha_2\lambda_6. \tag{47j}$$

As such the general solution of these ten coupled differential equations is a difficult task. Therefore, we resort to particular solutions of these differential equations and demonstrate the computation of the invariant for the case of equal mass and equal frequency, namely $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$. Further, we make an ansatz

$$\overset{\circ}{\lambda}_1 = \overset{\circ}{\lambda}_2 = -2\overset{\circ}{\psi}(t) \tag{48a}$$

leading to $\lambda_1 = -2\psi + c_1, \lambda_2 = -2\psi + c_2$. These choices immediately convert equations (47a)–(47e) to the forms

$$\alpha(\lambda_7 - \lambda_9) + \alpha_8(\lambda_8 - \lambda_{10}) = 0$$

$$\lambda_8 + \lambda_{10} = 2\bar{\alpha}_3\overset{\circ}{\psi} + \bar{\alpha}\overset{\circ}{\lambda}_3$$

$$\lambda_7 + \lambda_9 = -2\bar{\alpha}\overset{\circ}{\psi} - \bar{\alpha}_3\overset{\circ}{\lambda}_3$$

$$\overset{\circ}{\lambda}_4 = \left(\beta + \frac{\alpha\beta_3}{\alpha_3}\right)(\lambda_7 - \lambda_9) + 2\,\overset{\circ}{\psi}(\beta_3\bar{\alpha}_3 - \beta\bar{\alpha}) + (\bar{\alpha}\beta_3 - \bar{\alpha}_3\beta)\overset{\circ}{\lambda}_3$$
$$\overset{\circ}{\lambda}_5 = -\left(\beta + \frac{\alpha\beta_3}{\alpha_3}\right)(\lambda_7 - \lambda_9) + 2\,\overset{\circ}{\psi}(\beta_3\bar{\alpha}_3 - \beta\bar{\alpha}) + (\bar{\alpha}\beta_3 - \bar{\alpha}_3\beta)\overset{\circ}{\lambda}_3$$

where $\bar{\alpha} = \alpha/(\alpha_3^2 - \alpha^2)$; $\bar{\alpha}_3 = \alpha_3/(\alpha_3^2 - \alpha^2)$. An inspection of these results allows us to assume $\overset{\circ}{\lambda}_3 = 0$, i.e. λ_3 = constant c_3 (say) and $\lambda_7 = \lambda_9$ for simplicity. This implies a lot of simplification in the determination of the remaining λ's from equations (47*f*)–(47*j*). In particular, one immediately obtains

$$\overset{\circ}{\lambda}_4 = \overset{\circ}{\lambda}_5 = 2\,\overset{\circ}{\psi}(\beta_3\bar{\alpha}_3 - \beta\bar{\alpha}) \tag{48b}$$

leading to

$$\lambda_4 = \sigma_1(t) + c_4 \qquad \lambda_5 = \sigma_1(t) + c_5 \qquad \lambda_6 = \sigma_2(t) + c_6 \tag{48c}$$
$$\lambda_7 = \lambda_9 = -\bar{\alpha}\,\overset{\circ}{\psi} \qquad \lambda_8 = \lambda_{10} = \bar{\alpha}_3\,\overset{\circ}{\psi} \tag{48d}$$

where

$$\sigma_1(t) = 2\int \overset{\circ}{\psi}(\beta_3\bar{\alpha}_3 - \beta\bar{\alpha})\,\mathrm{d}t \qquad \sigma_2(t) = 2\int \overset{\circ}{\psi}(\beta\bar{\alpha}_3 - \beta_3\bar{\alpha})\,\mathrm{d}t \tag{48e}$$

and c_i $(i = 1, 2, \ldots, 6)$ are the constants of integration. In this way all ten λ_i are determined. Further we set $c_1 = c_2 = c_4 = c_5 = c_6 = 0$, and use these results for λ_i in (7) to obtain the final form of the invariant I, namely

$$I = -\psi(t)(p_1^2 + p_2^2) + \tfrac{1}{2}\sigma_1(t)(x_1^2 + x_2^2) + c_3 p_1 p_2 + \sigma_2(t)x_1x_2 \\ -\bar{\alpha}\,\overset{\circ}{\psi}(p_1x_1 + p_2x_2) + \bar{\alpha}_3\,\overset{\circ}{\psi}(p_1x_2 + p_2x_1) \tag{49}$$

for the system

$$H_1 = \tfrac{1}{2}\alpha(t)(p_1^2 + p_2^2) + \tfrac{1}{2}\beta(t)(x_1^2 + x_2^2) + \alpha_3(t)p_1p_2 + \beta_3(t)x_1x_2. \tag{50}$$

Here, $\psi(t)$ satisfies the constraining relations

$$\begin{aligned} \bar{\alpha}_3\overset{\circ\circ}{\psi} + \overset{\circ}{\bar{\alpha}}_3\,\overset{\circ}{\psi} &= -2\beta_3\psi + \beta c_3 - \alpha_3\sigma_1 - \alpha\sigma_2 \\ \bar{\alpha}\overset{\circ\circ}{\psi} + \overset{\circ}{\bar{\alpha}}\,\overset{\circ}{\psi} &= 2\beta\psi - \beta_3c_3 + \alpha\sigma_1 + \alpha_3\sigma_2. \end{aligned} \tag{51}$$

3.2. Invariant for H_2

To obtain the TD invariant for system (2), we make the following Γ_i and h_i identifications:

$$\begin{aligned} &\Gamma_1 = \tfrac{1}{2}p_1^2 \qquad \Gamma_2 = \tfrac{1}{2}p_2^2 \qquad \Gamma_3 = \tfrac{1}{2}x_1^2 \\ &\Gamma_4 = \tfrac{1}{2}x_2^2 \qquad \Gamma_5 = p_1x_2 \qquad \Gamma_6 = p_2x_1 \end{aligned} \tag{52a}$$

$$\begin{aligned} &h_1 = h_2 = \alpha(t) \qquad h_3 = h_4 = \beta(t) \\ &h_5 = -h_6 = f(t) \qquad \text{with} \quad \alpha(t) = 1/m \quad \beta(t) = m\omega^2. \end{aligned} \tag{52b}$$

The additional Γ_l needed to close the algebra are

$$\Gamma_7 = p_1p_2 \qquad \Gamma_8 = p_1x_1 \qquad \Gamma_9 = p_2x_2 \qquad \Gamma_{10} = x_1x_2 \tag{53}$$

with $h_7 = h_8 = h_9 = h_{10} = 0$. The number of nonvanishing Poisson brackets, in this case, turns out to be the same as for H_1 in section 3.1, namely

$$
\begin{array}{llll}
[\Gamma_1, \Gamma_3] = -\Gamma_8 & [\Gamma_1, \Gamma_6] = -\Gamma_7 & [\Gamma_1, \Gamma_8] = -2\Gamma_1 & [\Gamma_1, \Gamma_{10}] = -\Gamma_5 \\
[\Gamma_2, \Gamma_4] = -\Gamma_9 & [\Gamma_2, \Gamma_5] = -\Gamma_7 & [\Gamma_2, \Gamma_9] = -2\Gamma_2 & [\Gamma_2, \Gamma_{10}] = -\Gamma_6 \\
[\Gamma_3, \Gamma_5] = \Gamma_{10} & [\Gamma_3, \Gamma_7] = \Gamma_6 & [\Gamma_3, \Gamma_8] = 2\Gamma_3 & [\Gamma_4, \Gamma_6] = \Gamma_{10} \\
[\Gamma_4, \Gamma_7] = \Gamma_5 & [\Gamma_4, \Gamma_9] = 2\Gamma_4 & [\Gamma_5, \Gamma_6] = -\Gamma_9 + \Gamma_8 & \\
[\Gamma_5, \Gamma_7] = 2\Gamma_1 & [\Gamma_5, \Gamma_8] = -\Gamma_5 & [\Gamma_5, \Gamma_9] = \Gamma_5 & [\Gamma_5, \Gamma_{10}] = -2\Gamma_4 \\
[\Gamma_6, \Gamma_7] = 2\Gamma_2 & [\Gamma_6, \Gamma_8] = \Gamma_6 & [\Gamma_6, \Gamma_9] = -\Gamma_6 & [\Gamma_6, \Gamma_{10}] = -2\Gamma_3 \\
[\Gamma_7, \Gamma_8] = -\Gamma_7 & [\Gamma_7, \Gamma_9] = -\Gamma_7 & [\Gamma_7, \Gamma_{10}] = -\Gamma_8 - \Gamma_9 & \\
[\Gamma_8, \Gamma_{10}] = -\Gamma_{10} & [\Gamma_9, \Gamma_{10}] = -\Gamma_{10}. & &
\end{array}
\tag{54}
$$

As before, the substitution of these results in (8) for system (52) yields the following set of ten differential equations, namely

$$\overset{\circ}{\lambda}_1 = 2f\lambda_7 - 2\alpha\lambda_8 \tag{55a}$$
$$\overset{\circ}{\lambda}_2 = -2f\lambda_7 - 2\alpha\lambda_9 \tag{55b}$$
$$\overset{\circ}{\lambda}_3 = 2\beta\lambda_8 + 2f\lambda_{10} \tag{55c}$$
$$\overset{\circ}{\lambda}_4 = 2\beta\lambda_9 - 2f\lambda_{10} \tag{55d}$$
$$\overset{\circ}{\lambda}_5 = \beta\lambda_7 - f\lambda_8 + f\lambda_9 - \alpha\lambda_{10} \tag{55e}$$
$$\overset{\circ}{\lambda}_6 = \beta\lambda_7 - f\lambda_8 + f\lambda_9 - \alpha\lambda_{10} \tag{55f}$$
$$\overset{\circ}{\lambda}_7 = -f\lambda_1 + f\lambda_2 - \alpha\lambda_5 - \alpha\lambda_6 \tag{55g}$$
$$\overset{\circ}{\lambda}_8 = \beta\lambda_1 - \alpha\lambda_3 + f\lambda_5 + f\lambda_6 \tag{55h}$$
$$\overset{\circ}{\lambda}_9 = \beta\lambda_2 - \alpha\lambda_4 - f\lambda_5 - f\lambda_6 \tag{55i}$$
$$\overset{\circ}{\lambda}_{10} = -f\lambda_3 + f\lambda_4 + \beta\lambda_5 + \beta\lambda_6. \tag{55j}$$

As before, we look for the particular solutions of these coupled differential equations by making suitable choices for some of the λ_i. It can be seen that equations (55*e*) and (55*f*) are identical, immediately leading to

$$\lambda_5 = \eta(t) + c_5 \qquad \lambda_6 = \eta(t) + c_6 \tag{56}$$

where c_5 and c_6 are the constants of integration and $\overset{\circ}{\lambda}_5 = \overset{\circ}{\lambda}_6 = \overset{\circ}{\eta}$ (say). Further note from equations (55*a*)–(55*d*) and (55*g*)–(55*j*) that

$$\overset{\circ}{\lambda}_1 + \overset{\circ}{\lambda}_2 = -2\alpha(\lambda_8 + \lambda_9) \tag{57a}$$
$$\overset{\circ}{\lambda}_3 + \overset{\circ}{\lambda}_4 = 2\alpha(\lambda_8 + \lambda_9) \tag{57b}$$
$$\overset{\circ}{\lambda}_8 + \overset{\circ}{\lambda}_9 = \beta(\lambda_1 + \lambda_2) - \alpha(\lambda_3 + \lambda_4) \tag{57c}$$
$$\overset{\circ}{\lambda}_{10} + \frac{\beta}{\alpha}\overset{\circ}{\lambda}_7 = f\frac{\beta}{\alpha}(-\lambda_1 + \lambda_2) + f(-\lambda_3 + \lambda_4). \tag{57d}$$

In view of equations (57*c*) and (57*d*) we discuss the following two cases in terms of the ansatz for λ_i:

Case 1: Let

$$\lambda_1 = \lambda_2 = \psi(t) \qquad \lambda_3 = \lambda_4 = \rho(t)$$

then other λ_i can immediately be obtained as

$$\lambda_7 = -2\sigma_1 - \alpha(c_5 + c_6)t + c_7$$
$$\lambda_8 = \sigma_4 - \sigma_5 + 2\sigma_6 + (c_5 + c_6)\sigma_7 + c_8$$
$$\lambda_9 = \sigma_4 - \sigma_5 - 2\sigma_6 - (c_5 + c_6)\sigma_7 + c_9$$
$$\lambda_{10} = 2\sigma_2 + \sigma_3(c_5 + c_6) + c_{10}$$

where c_i are the constants of integration and σ_i are the time integrals given by

$$\begin{aligned} &\sigma_1(t) = \int \alpha\eta\,\mathrm{d}t \qquad \sigma_2(t) = \int \beta\eta\,\mathrm{d}t \qquad \sigma_3(t) = \int \beta\,\mathrm{d}t \qquad \sigma_4(t) = \int \beta\psi\,\mathrm{d}t \\ &\sigma_5(t) = \int \alpha\rho\,\mathrm{d}t \qquad \sigma_6(t) = \int f\eta\,\mathrm{d}t \qquad \sigma_7(t) = \int f\,\mathrm{d}t. \end{aligned} \tag{58a}$$

with a constraining relation

$$\alpha \overset{\circ}{\rho} + \beta \overset{\circ}{\psi} = 0. \tag{58b}$$

In this case a lot of simplification can be achieved if one assumes $\eta(t) = 0$ and $c_6 = -c_5$. This will lead to $\sigma_1 = \sigma_2 = \sigma_6 = 0$, $\lambda_7 = c_7$, $\lambda_{10} = c_{10}$, $\lambda_8 = \bar{\sigma} + c_8$, $\lambda_9 = \bar{\sigma} + c_9$ with $\bar{\sigma} = \sigma_4 - \sigma_5 = \int(\beta\psi - \alpha\rho)\,\mathrm{d}t$, and finally to the invariant (7) in the form

$$\begin{aligned} I = {}& \tfrac{1}{2}\psi(p_1^2 + p_2^2) + \tfrac{1}{2}\rho(x_1^2 + x_2^2) + \bar{\sigma}(p_1x_1 + p_2x_2) + c_5(p_1x_2 - p_2x_1) \\ &+ c_7p_1p_2 + c_8p_1x_1 + c_9p_2x_2 + c_{10}x_1x_2 \end{aligned} \tag{59}$$

for the system (2).

Case II. If we set

$$\lambda_1 = -\lambda_2 = \phi(t) \qquad \lambda_3 = -\lambda_4 = \chi(t) \qquad \lambda_8 = -\lambda_9 = \xi(t) \tag{60a}$$

in equations (57), then one can obtain the invariant without time integrals of the type (58*a*). In this case other λ_i from (55*g*)–(55*j*) turn out to be

$$\lambda_7 = (\overset{\circ}{\phi} + 2\alpha\xi)/2f \equiv a(t) \quad \text{(say)} \qquad \lambda_{10} = (\overset{\circ}{\chi} - 2\beta\xi)/2f \equiv b(t) \quad \text{(say)} \tag{60b}$$

the invariant can be obtained immediately from (7) as

$$\begin{aligned} I = {}& \tfrac{1}{2}\phi(p_1^2 - p_2^2) + \tfrac{1}{2}\chi(x_1^2 - x_2^2) + \eta(p_1x_2 + p_2x_1) + c_5(p_1x_2 - p_2x_1) \\ &+ a(t)p_1p_2 + \xi(p_1x_1 - p_2x_2) + b(t)x_1x_2 \end{aligned} \tag{61}$$

for the system (2) with the following constraining relations (cf equation (55*e*)):

$$\overset{\circ}{\eta} = \beta a(t) - 2f\xi - \alpha b(t) \qquad \overset{\circ}{a} = -2(f\phi + \alpha\eta) \qquad \overset{\circ}{b} = -2(f\chi - \beta\eta). \tag{62}$$

It may be remarked that for a suitable ansatz for λ_i this class of Hamiltonian systems along with the invariant (61) (or for that matter (59)) may constitute [23] the real and imaginary parts of a complex Hamiltonian in the spirit of the analyticity property of the latter.

To demonstrate the viability of the Lie algebraic method for a rather simpler TID version of H_2,we rewrite equations (55*a*)–(55*d*) as

$$fc_7 - \alpha c_8 = 0 \tag{63a}$$
$$fc_7 + \alpha c_9 = 0 \tag{63b}$$
$$\beta c_8 + fc_{10} = 0 \tag{63c}$$
$$\beta c_9 - fc_{10} = 0 \tag{63d}$$

rewrite equations (55*e*) and (55*f*) as

$$f(c_9 - c_8) - \alpha c_{10} + \beta c_7 = 0 \tag{63e}$$

and (55*g*)–(55*j*) as

$$f(-c_1 + c_2) - \alpha(c_5 + c_6) = 0 \tag{63f}$$

$$\beta(c_1 + c_2) - \alpha(c_3 + c_4) = 0 \tag{63g}$$

$$f(-c_3 + c_4) + \beta(c_5 + c_6) = 0 \tag{63h}$$

where $\lambda_m(t)$ in (55) are now replaced by constants c_m, $m = 1, \ldots, 10$, thereby reducing (8) to the form $\frac{dI}{dt} = [I, H] = 0$. The solution of equations (63*a*)–(63*d*) can be written immediately in terms of a constant k_1 as

$$c_8 = -c_9 = k_1 \qquad c_7 = \alpha k_1/f \qquad c_{10} = -\beta k_1/f \tag{64}$$

with a constraint

$$f^2 = \alpha\beta. \tag{65}$$

With regard to the solution of equations (63*e*)–(63*h*) two cases in terms of the setting of the constants c_i, $i = 1, \ldots, 6$, are possible. In the first case, if we set $c_1 = c_2 = k_2$, $c_5 = -c_6 = k_3$ which implies $c_3 = c_4 = (\beta k_2/\alpha)$, then the invariant from (7), after using (52) and (53), turns out to be

$$I = \frac{1}{2}k_2\left[p_1^2 + p_2^2 + \frac{\beta}{\alpha}(x_1^2 + x_2^2)\right] + k_3(p_1x_2 - p_2x_1) + k_1\left[p_1x_1 - p_2x_2 + \frac{1}{f}(\alpha p_1p_2 - \beta x_1x_2)\right]. \tag{66}$$

On the other hand, setting $c_1 = -c_2 = k_4$; $c_3 = -c_4 = \frac{\beta}{\alpha}k_4$, $c_6 = -\frac{2f}{\alpha}k_4 - c_5$ along with the results (64) yields the invariant (7) in the form

$$I = \frac{1}{2}k_4\left[p_1^2 - p_2^2 + \frac{\beta}{\alpha}(-x_1^2 + x_2^2) - \frac{4f}{\alpha}p_2x_1\right] + c_5(p_1x_2 - p_2x_1) + k_1\left[p_1x_1 - p_2x_2 + \frac{1}{f}(\alpha p_1p_2 - \beta x_1x_2)\right] \tag{67}$$

for the TID system (2). Note that each of the forms (66) and (67) of the TID invariant involves three arbitrary constants and the same exist with the common constraint (65) on the potential parameters. In other words, the invariant for the system H_2 exists in the resonance region particularly when equation (65) expressed as $f = \omega^2$ holds for the resonance since f is analogous to the forcing frequency and ω is the natural frequency.

4. Concluding discussion

After having investigated a more complicated class of Ermakov systems in three dimensions in [19], in this work we have once again exploited the rationalization and dynamical algebraic methods for the construction of exact invariants for the somewhat simpler two-dimensional coupled oscillator problem. As a matter of fact the systems ignored in the previous [19] work, particularly the ones involving both coordinate- and momentum-dependent couplings are investigated here with a possible generalization of the rationalization method to this effect. This is done to highlight the viability of these methods mainly for the case of momentum-dependent systems whose study in recent years has become desirable from the point of view of practical applications of these constructs in different contexts [9–14].

As pointed out in section 1, firstly the existence of an invariant for a dynamical system is questionable. If the invariant exists, then its construction, in general, is a difficult task. Once it is constructed and becomes available then not only its physical interpretation(s) but also its viability with regard to a better theoretical understanding of a given phenomenon is often a problem. In spite of all this, the availability of a few or all [4] invariants for a dynamical system definitely offers [3, 24] insight into the finer details as far as an understanding of the phenomenon is concerned. Often the applications of these constructs in different branches of theoretical science (see, e.g., [3] ch 7) are carried out on the basis of structural analogy [24] and at times by incorporating additional assumptions in the model concerned. With regard to the role of the constructs obtained in sections 2 and 3 in the study of coupled oscillator systems (1) and (2) the following remarks are in order:

(1) It may be noted that within the framework of the Lie algebraic approach, since the invariants are basically the superposition of a certain class of phase space functions (cf equation (7)), they can also offer alternative versions of the given Hamiltonian describing the same system but in totality. However, in these new equivalent forms of the Hamiltonian the individual terms may have different physical meanings from the point of view of couplings. Moreover, the standard structure of the Hamiltonian as 'the sum of kinetic and potential energy terms' often gets disturbed in these alternative versions but at the cost of a, perhaps simplified (!), description of the system with regard to its evolution in time. From this point of view the construction (61) (or for that matter (67)) for system (2) can be considered [25] as the Hamiltonian corresponding to indefinite kinetic energy and studied in the context of plasma physics. In fact, it has been argued [25] in the literature that only those two-dimensional Hamiltonians which deal with separable potentials give rise to positive definite kinetic energies. Clearly, H_2 of equation (2) is not such a choice. While our future studies will reveal more on this front, it appears now that certain features of the equations of motion (like linearity or nonlinearity) are retained when using the Lie algebraic approach through the closure property of the algebra. This, however, may not be the case in using the rationalization method, which otherwise appears to be more sound but considerably involved for the construction of dynamical invariants.
(2) In coupled mechanical oscillator problems, the transfer of energy from one oscillator to the other is attributed mainly to the x_1x_2-coupling term in the Hamiltonian. In the description using equivalent versions (invariants) these features might manifest through some other terms—maybe through the momentum-dependent ones. In the context of electrical circuits the role of invariant(s) can manifest through 'equivalent' circuit diagrams for which the analysis perhaps becomes easier (!). For example, the capacitor coupling in one case may become [9, 11] the inductor coupling in another case or vice versa, thereby suggesting the design of equivalent circuits for the same purpose.
(3) It is well known [26] that Bell's-inequality experiments are performed using harmonic oscillators since the photons, acting as oscillators, are excitations of modes of the underlying electromagnetic field. In particular, four harmonic oscillators having pairwise coupling at a time are considered to allow the exchange of energy leading to the preparation of a pair initially in an entangled state. Note that the TID versions of the invariant (cf equations (34) or (40)) particularly for system (1) could be of immediate concern in the analysis [10] of these experiments with a suitable choice of the arbitrary constants and couplings.
(4) Although TD systems now appear in different branches of mathematical sciences it may be of interest to investigate the role of the TD constructs of section 2 in the areas of lumped electrical circuits [11] and femtochemistry [27].

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